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SOME CHARACTERIZATIONS OF RIGHT (LEFT) WEAKLY REGULAR SEMIGROUPS

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Abstract. *In this paper, using the ideals theory, some characteriza- tions of three classes of semigroups are given: (1) right (left) weakly regular semigroups; (2) semigroups which are right (left) weakly regu- lar and left (right) duo-semigroups; (3) semigroups wich are right (left) weakly regular and intra regular. There are given also characterizations of right weakly regular and intra-regular elements*

Definition 1. *A semigroup S is called right (left) weakly regular if for every its right (left) ideal R (L)*

$$R^2 = R \; (L^2 = L)$$

A semigroup S is called weakly regular if it is right weakly regular and left weakly regular.

Theorem 1. *Every element a of a right (left) weakly regular semigroup S has the form*

$$a = ax \; (a = xa,) \text{ for any } x \in SaS.$$

Proof. Le a be an element of the right weakly semigroup S. It is clear that:

$$aS \subseteq a \cup aS = (a)_r.$$

Also:

$$(a)_r = (a)^2 = (a)_r (a)_r \subseteq (a)_r S = aS.$$

So we have that $(a)_r = aS$. Using this equality we find:

$$a \in (a)_r = (a)_r^2 = (aS)^2 = (aS)(aS) = a(SaS)$$

so $a = ax$ for the element $x \in SaS$.

Analoguously it can be shown that $(a)_l = Sa$ dhe $a = xa$ for any $x \in SaS$. □

Definition 2. *If every right (left) ideal of the semigroup S is a two-sided ideal of S than S is called a right (left) duo semigroup.*

Using definitions 1 and 2 we can prove the following:
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Theorem 2. *If the semigroup S is right (left) weakly regular and left (right) duo semigroup, than it is a regular semigroup.*

Proof. Let S be a semigroup satisfying the conditions of the theorem and let R, L be respectively a right and a left ideal of S. By Theorem 1, if

$$a \in R \cap L$$

than

$$a \in a(SaS) = (aS)aS \subseteq RLS$$

S is a left duo semigroup, so:

$$LS \subseteq L \ (*)$$

So, by $(*)$ we get subsequently:

$$a \in RLS \subseteq RL \ R \cap L \subseteq Rl.$$

The clear inclusion $RL \subseteq R \cap L$ yields

$$R \cap L = RL$$

So the semigroup S is a regular one. □

Definition 3. *A semigroup S is called **intra-regular** if every element $a \in S$ can be written as*

$$a = xa^2y \text{ for any } x, y \text{ in} S.$$

Theorem 3. *A semigroup is right weakly regular and intra-regular iff the inclusion*

$$R \cap B \subseteq RBR \quad (**)$$

holds for every right ideal R and every bi-ideal B such that

$$R \cap B \neq \emptyset.$$

Proof. Let S a right weakly regular and intra-regular semigroup. If $a \in S$, than there exist x, y, z, t from S such that:

$$a = axay = za^2t$$

If $a \in R \cap B$, where R is a right ideal and B a bi-ideal , than we have:

$$a = axay = axza^2ty = axayzaaty = (ax)(ayxza)(aty) \in RBR,$$

so

$$R \cap B \subseteq RBR$$

Now, let S be a semigroup satisfying the relation (∗∗).
If B = L, than we have

$$R \cap L \subseteq RLR \subseteq LR$$

for every right ideal R and every left ideal L of the semigroup S. So the semigroup S is intra-regular.
If we put B = R at (∗∗), we have:

$$R = R \cap R \subseteq RRR \subseteq R^2$$

The other inclusion $R^2 \subseteq R$ is obvious, so: $R^2 = R$ and the semigroup S is *right weakly regular*

□

Note. *(1) Theorem 3 holds even for any left weakly regular semigroup S such that*

$$L \cap B \subseteq LBL,$$

where L is a left ideal and B q bi-ideal.
(2) Theorem 3 is true even if the bi-ideal B is substituted by the quasi-ideal

Q.
In the following we give a characterization for the right weakly regular semi-groups.

Theorem 4. *The semigroup S right weakly regular iff*

$$R \cap B \cap I \subseteq RBI \quad (***)$$

for every right ideal R, every bi-ideal B and every two-sided ideal I such that

$$R \cap B \cap I \neq \emptyset$$

Proof. If S is a right weakly regular semigroup and $a \in R \cap B \cap I$, where R is a right ideal, B a bi-ideal and I a two-sided ideal, then there exist the elements x and y in S such that

$$a = axay = axaxayy = axaxaxayy = (ax)(axa)(xay^3) \in RBI$$

So the inclusion $(***)$ *holds.*

Vice versa.If S is a semigroup satisying the property $(***)$, one can write

$$R = R \cap R \cap S \subseteq RRS \subseteq R^2$$

for every right ideal R. By the obvious inclusion $R^2 \subseteq R$, *we get that*

$$R^2 = R.$$

So the semigroup S is right weakly regular. □

Note. (1) Theorem 4 has the following analoguous:
The semigroup S is a right weakly regular semigroup iff

$$L \cap B \cap I \subseteq IBL \ (***)$$

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for every left ideal L, for every bi-ideal B and for every two-sided bi-ideal I such that

$$L \cap B \cap I = \emptyset$$

Theorem 4. holds even if we substitute the bi-ideal B by the quasi-ideal

Q.

Definition 1. *The element a of the semigroup S is called weakly right (left) regular if* $a \in a(SaS)$ $(a \in (SaS)a)$

Taking into account that definition, the theorems 1 and 2 have the fol- lowiong analogues according the elements.

Theorem 5. *Let S be a semigroup. Than the following propositions are equiv- alent:*

1) The elemente $a \in S$ *is right weakly regular and intra-regular. 2)* $(a)_r \cap (a)_b \subseteq (a)_r\,(a)_b(a)_r$

Theorem 6. *The following propositions in a semigroup S are equivalent:*
1) The element $a \in S$ *is right weakly regular and intra-regular. 2)* $(a)_r \cap (a)_b \cap (a)_l \subseteq (a)_r\,(a)_b\,(a)_r\,(a)_l$

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